RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. THIRD SEMESTER EXAMINATION, DECEMBER 2017

SECOND YEAR [BATCH 2016-19]

Date : 12/12/2017 **MATHEMATICS [Honours]**

Time : 11 am – 3 pm **Paper : III** Full Marks : 100

[Use a separate Answer Book for each Group]

Group – A

***Answer any five questions from Question Nos. 1 to 8* :** [5×10]

1. a) Let V be an n-dimensional vector space over the field F and T be a linear operator on V such that the range and the null space of T are identical. Prove that n is even. Give an example of such a linear operator T. [2+3]

   b) Let $V = \mathbb{R}^3$ and W be a subspace of V generated by the vector (1, 0, 0). Find a basis of the quotient space V/W. Verify that $\dim V/W = \dim V - \dim W$. [2]

   c) Let V be the vector space of all $n \times n$ real matrices over $\mathbb{R}$ and $T: V \to V$ be a linear operator defined by $T(A) = \frac{A + A^t}{2}, A \in V$, where A^t denotes transpose of A. Find nullity of T. [3]

2. Let V and W be two vector spaces of dimensions n and m respectively over the same field F. Prove that L (V, W), the set of all linear transformations from V into W forms a vector space over the field F together with addition and scalar multiplications defined by $(T+U)(\alpha) = T\alpha + U\alpha$ and $(cT)(\alpha) = c(T\alpha)$ for all $T, U \in L(V, W)$ and $c \in F$.

   Also find the dimension of the space $L(V, W)$. [5+5]

3. a) Let V be an n-dimensional vector space over the field F and let $B = \{\alpha_1, ..., \alpha_n\}$ be an ordered basis for V.

      i) What is the matrix A of the linear operator T in the ordered basis B, defined by $T\alpha_j = \alpha_{j+1}$, $j = 1, ..., n-1$, $T\alpha_n = 0$? [3]

      ii) Prove that $T^n = O$ but $T^{n-1} \neq O$. [2]

   b) A linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x, y, z) = \left(x + z, \frac{5x - y + z}{2}, 3x + z\right) \forall (x, y, z) \in \mathbb{R}^3$. Find the matrix A of T relative to the ordered basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$. Deduce that T is invertible. Find T^{-1} and the matrix B of T^{-1} relative to the ordered basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$. Verify that $B = A^{-1}$. [1+1+1+1+1]

4. a) Let V and W be two finite dimensional vector spaces over the same field F. Prove that V and W are isomorphic if and only if dim V = dim W. [5]

   b) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ and $U: \mathbb{R}^2 \to \mathbb{R}^3$ be two linear transformations. Prove that UT is not invertible. [2]

   c) Give examples of two matrices which have same characteristic polynomial but are not similar. Justification needed. [3]

5. a) Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix, where $A=\begin{pmatrix}1 & -2 & 0\\ -2 & 2 & -2\\ 0 & -2 & 3\end{pmatrix}$. [5]

b) Determine the conditions for which the system of equations

$$x+y+z=b$$
$$2x+y+3z=b+1$$
$$5x+2y+az=b^2$$

has (i) only one solution, (ii) no solution, (iii) many solutions. [5]

6. a) Let V be a finite dimensional inner product space and f be a linear functional on V. Prove that there exists a unique vector β in V such that $f(\alpha)=(\alpha\,|\,\beta)$ for all α in V. [4]

b) Let V be a finite dimensional inner product space and T a linear operator on V. Show that range of T* is the orthogonal complement of the null space of T, where T* is the adjoint of T. [3]

c) Let V be the inner product space $C([0,1])$ over □ with the inner product $<f,g>=\int_0^1 f(t)g(t)dt$, where $C([0,1])$ is the set of all continuous real valued maps defined on [0,1]. Let W be the subspace of V spanned by the linearly independent set $\beta=\{t,\sqrt{t}\}$. Using Gram-Schmidt process find an orthonormal basis for W from the basis β of W. [3]

7. a) The matrix of $T:□^2\to□^2$ is given by $A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ relative to the standard ordered basis of $□^2$. Find T and T* and verify whether T is a normal operator, where T* is the adjoint of T. [2+2+1]

b) Let V be a finite dimensional inner product space over F(= □ or □) and let T be an unitary operator on V. Suppose $\lambda\in F$ be an eigen value of T. Show that $|\lambda|=1$. [2]

c) Let $T:□^2\to□^2$ be the reflection of the points through the line y = – x. Find the matrix of T relative to the standard ordered basis of $□^2$. [3]

8. a) Reduce the quadratic form : $2x^2+5y^2+10z^2+4xy+12yz+6zx$ to its normal form. Find also the rank and the signature. [5]

b) Let A be any square matrix over □ and its characteristic equation is $x^2-x+1=0$. Find det A and Trace A. [1+1]

c) Let V be an inner product space and T be a self-adjoint linear operator on V. Prove that each characteristic value of T is real and characteristic vectors of T associated with distinct characteristic values are orthogonal. [3]

<u>Group – B</u>

Answer <u>any four</u> questions from <u>Question Nos. 9 to 14</u> **:** [4×5]

9. The plane $ax+by+cz+d=0$ bisects an angle between a pair of planes one of which is $\ell x+my+nz=p$. Show that the equation of the other plane is $(a^2+b^2+c^2)(\ell x+my+nz-p)=2(a\ell+bm+cn)(ax+by+cz+d)$. [5]

10. Prove that the line of shortest distance between z-axis and the variable line $\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right)$, $\frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right)$, (where λ varies) generates the surface $abz(x^2+y^2)=(a^2-b^2)cxy$. [5]

11. a) Write down the law of transformation if the coordinate axes are rotated about the origin such that the new axes have the following direction ratios : $-\frac{b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}, 0$; $-\frac{ac}{\sqrt{a^2+b^2}}, -\frac{bc}{\sqrt{a^2+b^2}}, \sqrt{a^2+b^2}$ and a, b, c. [1]

b) If a plane intersects a sphere in more than one point, then prove that its intersection with the sphere is a circle. [4]

12. Prove that the planes, which cuts the cone $ax^2+by^2+cz^2=0$ in perpendicular straight lines, touch the cone $\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0$. [5]

13. The normal at a point P of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets the yz-plane and the zx-plane in G and G' respectively. OQ is drawn from the origin O making equal and parallel to GG'. Prove that the locus of Q is the ellipsoid $a^2x^2+b^2y^2+c^2z^2=(a^2-b^2)^2$. [5]

14. Show that the equation of the generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ through a point $(0, b\sec\theta, c\tan\theta)$ of the hyperbolic section of the hyperboloid by the co-ordinate plane x = 0 are $\frac{x}{\pm a}=\frac{y-b\sec\theta}{b\tan\theta}=\frac{z-c\tan\theta}{c\sec\theta}$. [5]

Answer any three questions from Question Nos. 15 to 19 : [3×10]

15. a) A particle moves in a straight line with an acceleration towards a fixed point in the straight line, which is equal to $\frac{\mu}{x^2}-\frac{\lambda}{x^3}$ when the particle is at a distance x from the given point. If it starts from rest at a distance 'a' from the fixed point then show that it oscillates between this distance and the distance $\frac{\lambda a}{2\mu a-\lambda}$ and that its periodic time is $\frac{2\pi\mu a^3}{(2a\mu-\lambda)^{3/2}}$. [7]

b) For a particle moving on a smooth plane curve under the action of conservative forces, show that the change in kinetic energy is equal to work done by the forces. [3]

16. a) Find the components of velocity and acceleration of a moving point referred to a set of rectangular axes revolving with uniform angular velocity ω about the origin in their own plane. [6]

b) A particle is projected with velocity V from a cusp of a smooth cycloid with axis vertical and vertex downwards, down the arc. Find its time of reaching the vertex. [4]

17. a) A particle moves in a straight line with acceleration n^2x (distance) towards a fixed point in the line and with an additional periodic push $L\cos(pt)$ [L, n, p are constants]. If the particle starts from rest at a distance 'a' from the centre, find its distance from the fixed point in terms of time. [5]

b) A particle of mass m is falling under the influence of gravity through a medium whose resistance equal to μ times the velocity. If the particle be released from rest, then show that the distance fallen through in time t is $g\frac{m^2}{\mu^2}\left(e^{-\frac{\mu}{m}t}-1+\frac{\mu}{m}t\right)$. [5]

18. a) A particle describes a path which is nearly a circle under the action of a central force at the centre of the circle, varying inversely as the nth power of its distance from the centre. Find the condition that the motion may be stable. [6]

b) A particle describes an ellipse under a force which is always directed towards the centre of the ellipse. Find the law of force. [4]

19. a) A particle is projected from an apse at a distance 'a' with a velocity from infinity under the action of a central acceleration $\frac{\mu}{r^{2n+3}}$. Prove that the path is $r^n = a^n \cos(n\theta)$. [7]

b) A particle describes the curve $r = e^{\theta}$ in such a manner that its radial acceleration is zero. Prove that its angular velocity is constant. [3]

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